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Interaction Stability Analysis from the Input-Output Viewpoints

Yuancan Huang, *Senior Member, IEEE,* and Qiang Huang, *Fellow, IEEE*

Abstract— **Interaction with the environment is arguably one of the necessary actions for many robot applications such as haptic devices, manipulation, parts assembly, cooperation with humans, and the use of tools. Taxonomy of interaction behaviours is classified into three categories: cooperation, collaboration, and competition. In theory, interaction dynamics may be modelled by D'Alembert's principle or nonsmooth mechanics through seeking equality and/or inequality kinematic constraints. However, it is hard to gain these kinematic constraints in practice since they may be variable or be hardly described in a mathematical form. As a result, bond graph methodology is preferred in interaction dynamics modelling.**

In this paper, passivity and passivity indices with the differential operator are put forward by restricting its domain from the whole extended Hilbert function space to a set of all continuous function with finite derivative, and then the input-output stability condition, in this case, is derived. Next, mechanical impedance and admittance are defined, and a linear spatial impedance representation is given from the energetic point of view. Base on the bond graph theory, an ideal model is presented to model the idealized interaction, and invariance of port functions derived from the ideal interaction model is introduced; An interaction model is then proposed accounting for nonidealized factors and to describe cooperative, collaborative, and competitive interactions in a unified way. Finally, interaction stabilities are analyzed corresponding to different interaction models, and robustness of interaction stability is addressed based on the passivity indices.

I. Introduction

Robot interaction implicates narrowly that a robot comes into contact with a human being or a mechanical system to execute an interactive task [1]. Taxonomy of interaction behaviours is classified into three categories: cooperation, collaboration and competition [2]. Due to the versatility of interaction, kinematic constraints may be embodied in any form, holonomic, nonholonomic, scleronomic, rheonomic, bilateral, or unilateral. Of course, when the equality and/or inequality kinematic constraints are mathematically described, interaction dynamics may be derived from D'Alembert's principle [3], [4] or nonsmooth mechanics [5]. However, kinematic constraints in many interactive tasks are usually hard to be described in a mathematical form, *e.g.*, imagine that a robot arm gives humans a handshake or a hand wrestle. Alternatively, interaction dynamics may be modelled via inspecting the energy transaction between a robot and the environment. This energy-based modelling approach is called bond graphs [6], [7], which is a graphical description formalism particularly suited for multidisciplinary or mechatronic systems devised by H. Paynter in as early as 1959. Nowadays, this modelling methodology is worldwide accepted not only in research and education but also in industry [8].

N. Hogan, a pioneer in impedance control, explored interaction dynamics based on the bond graph theory and introduced the common velocity postulate in interaction model [9], [12]. Interaction stability was analyzed based on passive network theory and passivity theorem [14], [15], [11], and the limitations on impedance control was probed in [16], [17], [18]. The nonlinear equivalent networks were suggested to model variable-impedance actuators in [13]. In [19], [20], water effect in impedance control was studied by using generalized Bode sensitivity integral on the general control system [21]. With an understanding of the conservativeness of passivity condition, a μ-based impedance controller was proposed by using small-gain theorem through building the parameter-perturbed environment model and introducing complementary stability concept [22].

As a matter of fact, the bond-graph-based interaction model subsumes the general input-output systems intensively studied in control theory [23], [24], [27]. However, this point seems not well-explored in the existing literature. In this paper, we study interaction stability from the input-output viewpoint in control theory and address the following critical issues concerning interaction: *i)* What are the input-output stability conditions while the differential operator is included? *ii)* under what conditions, the ideal interaction model is validated and invariance of port functions of two interacted subsystems is granted; *iii)* the idea of modelling uncertainties in robust control is borrowed to deal with nonidealized factors in interaction, and a generic framework is proposed to describe cooperative, collaborative, and competitive interactions in a unified way; and *iv)* how to characterize robustness of interaction stability.

In the following, 'force' implies 'force/torque' and 'velocity' 'linear velocity/angular velocity' unless specified in the context.

II. Notations, Notions and Preliminaries Results

A. Notations

$\mathscr{L}_2(\mathbb{R}^n)$ can be thought of as the space of $\mathbb{R}^n$-valued function $f:\,[0,\infty)\rightarrow\mathbb{R}^n$ of finite energy $\|f\|^2=\int_0^\infty|f(t)|^2dt$. This is a subset of the *extended space* $\mathscr{L}_{2e}(\mathbb{R}^n)$, whose

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The authors are all with the Beijing Advanced Innovation Center for Intelligent Robots and Systems, School of Mechatronical Engineering, Beijing Institute of Technology, No.5 South Zhongguancun Street, Haidian District, 100081 Beijing, China (e-mail: yuancanhuang@bit.edu.cn).

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members only need to be square integral on finite intervals. The *inner product* of signals y and u in $\mathscr{L}_2(\mathbb{R}^n)$ is denoted by $\langle y,u\rangle = \int_0^\infty y^T(t)u(t)dt$, whose *induced norm* is $\|z\| = \sqrt{\langle z,z\rangle}$. *Hilbert space*, $\mathscr{H}$, is an inner product space that is complete as a normed linear space under the induced norm. Similarly, the *inner product* of signals y and u in $\mathscr{L}_{2e}(\mathbb{R}^n)$ over the interval $[0,T]$ is $\langle y,u\rangle_T = \int_0^T y^T(t)u(t)dt$. The *extended Hilbert space*, $\mathscr{H}_e$, is the space of all $z \in \mathscr{L}_{2e}(\mathbb{R}^n)$ with the property that $\|z\|_T^2 = \langle z,z\rangle_T$ is finite for all $T \geq 0$.

B. The differential operator

Definition 2.1: An *operator* in $\mathscr{H}_e$ is a function $\Gamma: \mathscr{D} \subset \mathscr{H}_e \to \mathscr{H}_e$. The set $\mathscr{D}$ is called the *domain* $\mathscr{D}(\Gamma)$ of Γ. The set of all values $\Gamma(u)$, $u \in \mathscr{D}(\Gamma)$, is called the *range*, $\mathscr{R}(\Gamma)$, of Γ. The set of all elements $(u,\Gamma(u))$, $u \in \mathscr{D}(\Gamma)$ and $\Gamma(u) \in \mathscr{R}(\Gamma)$, is called the *graph*, $Gr(\Gamma)$, of Γ. Hence $\mathscr{D}(\Gamma) \subset \mathscr{H}_e$, $\mathscr{R}(\Gamma) \subset \mathscr{H}_e$ and $Gr(\Gamma) \subset \mathscr{H}_e \times \mathscr{H}_e$.

The following concepts describe the properties of an operator [25], [26]:

- The operator Γ is called *bounded* if $\Gamma(\mathscr{B})$ is a bounded set in $\mathscr{H}_e$ for any bounded set $\mathscr{B} \subset \mathscr{H}_e$. Otherwise, it is called unbounded.
- The operator Γ is called *single-valued* if there is associated exactly one value $\Gamma(u)$ with each element u in $\mathscr{D}(\Gamma)$.
- The operator Γ is said to be *causal* (or, equivalently, *nonanticipative*) if and only if $P_T(u) = P_T(v)$ implies that $P_T\Gamma(u) = P_T\Gamma(v)$, $\forall\, u,v \in \mathscr{H}_e$, $\forall T \geq 0$, where P_T is the *past projection* (truncation), which leaves a function unchanged on the interval $[0,T]$ and gives the value zero on (T,∞). Thus, for causal operators, past outputs depend only on past inputs.
- The operator Γ is called *linear* if $Gr(\Gamma)$ is a linear space. Since every linear set contains the zero element, $(0,0)$ is in the graph of every linear operator. A necessary and sufficient condition that a linear operator Γ be single-valued is that $\Gamma(0)$ has the unique value 0.
- The operator Γ is said to be $\mathscr{L}_2$*-stable* if $u \in \mathscr{H} \Rightarrow \Gamma(u) \in \mathscr{H}$.
- The operator Γ is said to have *finite* $\mathscr{L}_2$*-gain* if there exist finite constants $\gamma > 0$ and b such that for all $T \geq 0$

$$\|(\Gamma(u))_T\| \leq \gamma\|u_T\| + b, \quad \forall u \in \mathscr{H}_e \tag{1}$$

  Obviously, b can be assumed to be nonnegative without loss of generality. Furthermore, Γ is said to have finite gain with zero bias if $b = 0$.

When we consider an impedance or admittance operator, the differential operator, $\frac{d}{dt}$, usually comes into existence, *e.g.*, the impedance of a pure mass or the admittance of a pure spring. First, the differential operator is ill-defined in the whole space $\mathscr{H}_e$, for the derivative of a function with finite jump discontinuities is infinite. Second, we have

Lemma 2.2: The differential operator $\frac{d}{dt}: \mathscr{A} \subset \mathscr{H}_e \to \mathscr{H}_e$ where $\mathscr{A}$ denotes a set of all continuous functions with finite derivative is an unbounded, noncausal, linear operator.

Proof: The linearity is straightforward. Since the differentiation operation takes the bounded set $\{t \mapsto \sin(nt)\}_{n\in\mathscr{N}}$ to the unbounded set $\{t \mapsto n\cos(nt)\}_{n\in\mathscr{N}}$, it is unbounded by definition. Obviously, the differential operator does not satisfy the causality definition. ■

C. Passivity and passivity Indices with the differential operator

Consider an operator $\Gamma: \mathscr{H}_e \to \mathscr{H}_e$ which is well-defined. It is called

- *passive* if there exists constant β such that

$$\langle\Gamma(u),u\rangle_T \geq -\beta, \quad \forall u \in \mathscr{H}_e,\, T \geq 0 \tag{2}$$

- *strictly input passive* if there exist constants $\delta > 0$ and β such that

$$\langle\Gamma(u),u\rangle_T \geq \delta\|u_T\|^2 - \beta, \quad \forall u \in \mathscr{H}_e,\, T \geq 0 \tag{3}$$

- *strictly output passive* if there exist constants $\varepsilon > 0$ and β such that

$$\langle\Gamma(u),u\rangle_T \geq \varepsilon\|(\Gamma(u))_T\|^2 - \beta, \quad \forall u \in \mathscr{H}_e,\, T \geq 0 \tag{4}$$

Obviously, β can also be assumed to be nonnegative without loss of generality.

Lemma 2.3: Assume that an operator Γ is single-valued and that the inverse operation leads to the nonincreasing initial energy. The following results hold:

i). Γ is passive if and only if Γ^{-1} is passive.

ii). Γ is strictly output passive (or, strictly input passive) if and only if Γ^{-1} is strictly input passive (or, strictly output passive).

Proof: Since the operator Γ is single-valued, its inverse Γ^{-1} is well-defined. Together with the non-increasing initial energy property of the inverse operations, all the results are readily derived from the above definitions. ■

In order to assess the degree of passivity for both passive and nonpassive systems, passivity indices is defined to measure an excess or shortage of passivity [28], [29].

Definition 2.4: Consider the feedfoward and feedback configurations in Figs. 1. If the feedforward-connected operator $\tilde{\Gamma}_{ff}$ is passive, then Γ is called *input feedforward passive with passivity index* $\nu \in \mathbb{R}$, denoted by IFP(ν); And, if the feedback-connected operator $\tilde{\Gamma}_{fb}$ is passive, then Γ is called *output feedback passive with passivity index* $\rho \in \mathbb{R}$, denoted by OFP(ρ).

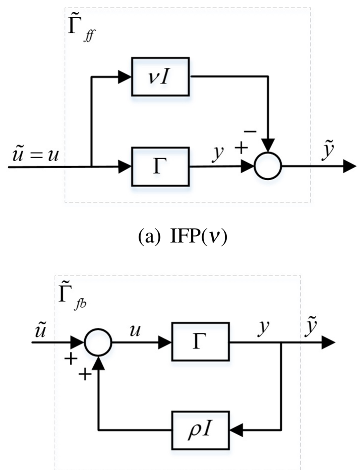
When $\nu = \rho = 0$, IFP(ν) and OFP(ρ) degenerate into passivity. When $\nu > 0$, $\rho > 0$, IFP(ν) and OFP(ρ) implicit that Γ has an excess of passivity. In this case, Γ is strictly input passive or strictly output passive. When $\nu < 0$, $\rho < 0$, Γ is nonpassive.

The following properties are straightforward:

- If Γ is input feedforward passive with some passivity index ν_0, then it has IFP($\nu \leq \nu_0$).
- If Γ is output feedback passive with some passivity index ρ_0, then it has OFP($\rho \leq \rho_0$).

Similarly, we can draw a conclusion that

Lemma 2.5: Assume that an operator Γ is single-valued and that there are no initial energy increments in inverse and



(a) IFP(ν)

(b) OFP(ρ)

Fig. 1: Excess and shortage of passivity.

feedback operations. Then, Γ has IFP(ν) (or, OFP(ρ)) if and only if Γ^{-1} has OFP(ν) (or, IFP(ρ)).

D. $\mathscr{L}_2$-stability with the differential operator

We wonder whether or not the input-output stability theorems in [27] still hold, provided that the differential operator is included.

Theorem 2.6: Let $\Gamma : \mathscr{H}_e \to \mathscr{H}_e$ which is well-defined. If Γ has finite $\mathscr{L}_2$-gain, then it is $\mathscr{L}_2$-stable. But the converse may not be true.

Proof: Assuming $u \in \mathscr{H}$ and letting $T \to \infty$ in (1) we obtain

$$\|\Gamma(u)\| \leq \gamma\|u\| + b, \quad \forall u \in \mathscr{H} \tag{5}$$

which implies that $\Gamma(u) \in \mathscr{H}$ for all $u \in \mathscr{H}$.

Since $P_T\Gamma = P_T\Gamma P_T$ may no longer hold due to the differential operator, the converse is not always true. ■

Now, we state the relationship between passivity and stability.

Theorem 2.7: Let $\Gamma : \mathscr{H}_e \to \mathscr{H}_e$ which is well-defined be strictly output passive. Then Γ is $\mathscr{L}_2$-stable.

Proof: Let $y = \Gamma(u)$. Since Γ is strictly output passive, there exist $\varepsilon > 0$ and β such that

$$\begin{aligned} \varepsilon\|y_T\|^2 &\leq \langle y,u\rangle_T + \beta \\ &\leq \langle y,u\rangle_T + \beta + \frac{1}{2}\left\|\frac{1}{\sqrt{\varepsilon}}u_T - \sqrt{\varepsilon}y_T\right\|^2 \\ &= \beta + \frac{1}{2\varepsilon}\|u_T\|^2 + \frac{\varepsilon}{2}\|y_T\|^2 \end{aligned} \tag{6}$$

By the last inequality, we have

$$\|y_T\|^2 \leq \frac{1}{\varepsilon^2}\|u_T\|^2 + \frac{2\beta}{\varepsilon} \tag{7}$$

which implies that Γ has finite gain $\leq \frac{1}{\varepsilon}$. We conclude the result from the above theorem. ■

III. Mechanical impedance and admittance

A. Definitions

Causality implies that, of two variables in a bond, one is independent and the other is dependent. It is imperative that a direction of causality may be imparted to an energetic exchange since no quantitative analysis of any form is possible until this is done. For a mechanical system, assuming force is analogous to voltage and velocity to current, mechanical impedance and admittance are defined by analogy with their electrical counterparts.

Definition 3.1: Mechanical impedance at a port, denoted Z, is a dynamic operator that determines an output force time function from an input velocity time function at the same port. Mechanical admittance at a port, denoted Y, is a dynamic operator that determines an output velocity time function from a force time function at the same port.

If the system is linear and time-invaraint, admittance is the inverse of impedance, and both can be represented in the Laplace domain by transfer functions, $Z(s)$ or $Y(s)$.

Of course, impedance and admittance can be generalized to nonlinear, time-varying systems. Mechanical impedance may be described by state and output equations relating the input velocity, V, to the output force, F, as follows:

$$\begin{aligned} \dot{x} &= f(x,V,t) \\ F &= h(x,V,t) \end{aligned} \tag{8}$$

where x is a finite-dimensional vector of state variables and $P = F^TV$ represents the instantaneous power transferred between the system and its environment.

In the nonlinear case, mechanical admittance is the causal dual of impedance in that the roles of input and output are exchanged; mechanical admittance may be described by state and output equations relating input force to output velocity. However, as the required inverse may not be definable, mechanical admittance may not be the inverse of mechanical impedance and *vice versa*.

B. A linear spatial impedance representation [30]

In robotics, the configuration of the robot end-effector may be identified by a 3-dimensional position vector p_e and a 3×3 rotation matrix R_e that represent the origin and orientation of a frame attached to the robot end-effector relative to a base frame. When the robot end-effector interacts with the environment, we introduce a virtual frame specified by p_v and R_v identifying the *virtual equilibrium configuration* such that a mechanical impedance can be defined in terms of displacements between the two frames.

The end-effector velocity is given by the linear velocity vector $\dot{p}_e$ and the angular velocity vector ω_e where

$$\dot{R}_e = S(\omega_e)R_e = R_eS(\omega_e^e) \tag{9}$$

being $S(\cdot)$ the skew-symmetric matrix operator performing vector product. As usual, the superscript denotes the frame to which a quantity (vector or matrix) is referred; the superscript is dropped whenever a quantity is referred to the base frame. Also, the end-effector linear and angular acceleration are respectively given by $\ddot{p}_e$ and $\dot{\omega}_e$.

Likewise, the velocity and acceleration of the virtual frame are respectively given by $\dot p_v$, ω_v and $\ddot p_v$, $\dot\omega_v$.

The translation and rotation displacements between the end-effector frame and the virtual frame can be characterized by the position error

$$\tilde p^v_{ev} = p^v_e - p^v_v = R_v^T(p_e - p_v) \tag{10}$$

and the orientation error

$$\tilde R^v_e = R_v^T R_e \tag{11}$$

By using an angle/axis description, $\tilde R^v_e$ can be generated by the rotation of an angle θ_{ev} about an axis of unit vector r^v. Hence, the orientation error can be expressed by a unitary quaternion:

$$\begin{aligned}\eta_{ev} &= \cos\frac{\theta_{ev}}{2}\\ \varepsilon^v_{ev} &= \sin\frac{\theta_{ev}}{2} r^v\end{aligned} \tag{12}$$

When $\eta_{ev} = \pm 1$ and $\varepsilon^e_{ev} = 0$, the end-effector frame is aligned with the virtual frame.

The relationship between the time derivative of the quaternion is established by the so-called quaternion propagation:

$$\dot\eta_{ev} = -\frac{1}{2}(\varepsilon^v_{ev})^T\tilde\omega^v_{ev} \tag{13a}$$

$$\dot\varepsilon^v_{ev} = \frac{1}{2}E(\eta_{ev},\varepsilon^v_{ev})\tilde\omega^v_{ev} \tag{13b}$$

where $\tilde\omega^v_{ev} = \omega^v_e - \omega^v_v = R_v^T(\omega_e - \omega_v)$ is the angular velocity error and $E = \eta_{ev}I + S(\tilde\varepsilon^v_{ev})$.

Define the kinetic energy

$$\mathscr{T} = \frac{1}{2}(\dot{\tilde p}^v_{ev})^T M_p(p_e)\dot{\tilde p}^v_{ev} + \frac{1}{2}(\tilde\omega^v_{ev})^T M_o(p_e)\tilde\omega^v_{ev} \tag{14}$$

where $M_p(p_e)$ and $M_o(p_e)$ are symmetric positive definite matrices. In particular, when $M_p(p_e) = m_p I$, the first term expresses the translational kinetic energy of a rigid body with mass m_p whose center of mass has a linear velocity $\dot{\tilde p}_{ev}$; otherwise, in the general case, $M_p(p_e)$ can be regarded as a generalized mass. On the other hand, the second term expresses the rotational kinetic energy of a rigid body with inertia tensor $M_o(p_e)$ and angular velocity $\tilde\omega_{ev}$.

Now, consider the potential elastic energy

$$\mathscr{U} = \frac{1}{2}(\tilde p^v_{ev})^T K_p\tilde p^v_{ev} + 2(\varepsilon^v_{ev})^T K_o\varepsilon^v_{ev} + 2\eta_{ev}(\varepsilon^v_{ev})^T K_m\tilde p^v_{ev} \tag{15}$$

where K_p and K_o are constant symmetric positive definite matrices. Select a constant matrix K_m such that the equivalent stiffness matrix

$$K = \begin{bmatrix} K_p & 2K_m^T \\ 2K_m & 4K_o \end{bmatrix}$$

is positive definite. A sufficient condition for the positive definiteness of K is

$$\lambda_{min}(K_p)\lambda_{min}(K_o) > \sigma^2_{max}(K_m) \tag{16}$$

where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue and $\sigma_{max}(\cdot)$ the maximum singular value.

The first term on the right-hand side of (15) expresses the elastic energy stored in a three-dimensional linear spring with stiffness matrix K_p. The second term is equivalent to the rotational elastic energy stored in a three-dimensional torsional spring of stiffness matrix K_o acting as to align frame R_e with R_v. When K_m is symmetric and higher order terms are neglected, the last term coincides with the coupling elastic energy.

A mechanical impedance at the end effector can be defined in terms of its translational part and its rotational part [31], [32]:

$$\begin{aligned} M_p(p_e)\ddot{\tilde p}^v_{ev} + D_p\dot{\tilde p}^v_{ev} + K_p\tilde p^v_{ev} + 2\eta_{ev}K_m^T\tilde\omega^v_{ev} &= f^v \\ M_o(p_e)\dot{\tilde\omega}^v_{ev} + D_o\tilde\omega^v_{ev} + \bar K_m\tilde p^v_{ev} + \bar K_o\varepsilon^v_{ev} &= \mu^v \end{aligned} \tag{17}$$

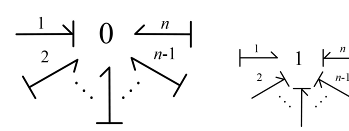
where D_p and D_o are symmetric positive definite constant matrices characterizing a translational and rotational damping at the end effector, respectively, (f^v, μ^v) is the wrench applied at the end effector, and

$$\begin{aligned}\bar K_m &= 2E^T(\eta_{ev},\varepsilon^v_{ev})K_o \\ \bar K_o &= [\eta_{ev}E(\eta_{ev},\varepsilon^v_{ev}) - \varepsilon^v_{ev}(\varepsilon^v_{ev})^T]^T K_m\end{aligned}$$

IV. Interaction Dynamics Modelling and Stability Analysis

1) Ideal junctions: An *ideal junction* transmits power instantaneously between its ports, without storing or dissipating energy; it is nonenergic. An important fact concerning ideal junctions has been shown that there are only two possible continuous, port-symmetric, power-conserving, nonenergic, 3-port junctions: a *common effort* or *type zero* junction and a *common flow* or *type one* junction. In other words, three subsystems are obliged to possess either a common effort or a common flow while they are bonded with the ideal three-port junctions [10].

Bond graphs of the n-port type-zero and type-one junctions with causality assignment are pictured in Fig. 2, where the half-arrow sign convention determines the assumed direction of positive energy flow and the causal strokes indicate the fact that one and only one subsystem may determine the flow at a 1-junction; one and only one subsystem may determine the effort at a 0-junction.



(a) 0-junction (b) 1-junction

Fig. 2: The n-port 0- and 1-junctions with causality assignment.

Based on the continuity of power [6], for a 0-junction, the generalized Kirchhoff's node law holds:

$$\begin{aligned} e_i &= e_1, \quad i = 2,\dots,n \\ \sum_{i=1}^{n} f_i &= 0 \end{aligned} \tag{18}$$

and, for a 1-junction, the generalized Kirchhoff's loop law holds:

$$\begin{aligned} f_i &= f_1, \quad i = 2, \ldots, n \\ \sum_{i=1}^{n} e_i &= 0 \end{aligned} \tag{19}$$

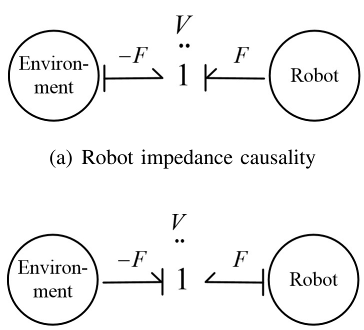
In mechanical engineering, 0-junction represents a geometric compatibility for a situation involving a force and velocities which algebraically sums to zero; 1-junction represents a dynamic equilibrium of forces associated with a common velocity [7].

A. Interaction models

1) Ideal interaction model [9]: While a robot and the environment is interconnected with the ideal 3-port 1-junction, the generalized Kirchhoff loop law guarantees that

$$F_r = -F_e, \quad V_r = V_e \tag{20}$$

which complies with the Newton's action and reaction law and satisfies the common velocity postulate. The bond graphs for impedance control and admittance control are respectively plotted in Figs. 3.



(a) Robot impedance causality

(b) Robot admittance causality

Fig. 3: The ideal interaction models.

Obviously, we can deduce the following result from Definition 3.1 and (20):

Theorem 4.1 (Invariance of port functions): Under the common velocity postulate, assume that that the virtual equilibrium configurations of two interacted subsystems are same or are kept constant in interaction, port impedance and port admittance remain invariant while the two subsystems are coupled.

Remark 4.2: This invariance property guarantees that interaction behaviours may be regulated through manipulating port functions of two interacted subsystems.

2) Interaction model: The ideal interaction model (20) portrays the ideal interaction process which can never occur in a real world. First of all, it is impossible to transact power between physically distinct locations instantaneously. Second, many factors may result in force errors and velocity errors. For examples, force errors may be incurred by neglected flexibility, damping and mass in parallel with one of interacted subsystem or by neglected gravity; velocity errors from distinct virtual equilibrium configurations and/or dislocation of contact positions and target positions, or from neglected damping in series with one of interacted subsystem. Moreover, lumped-parameter reduction of continuous structures will inevitably incur both force errors and velocity errors.

Similar to uncertainty modelling philosophy in robust control theory, if the ideal interaction model is regarded as a nominal model, then the force errors and the velocity errors may be thought as the extraneous disturbances. Therefore, an interaction model to account for nonidealized factors is equivalent to a feedback connection of impedance and admittance in the signal-flow diagram, as shown in Fig. 4 where F denotes the contact forces, V the common velocities, and Z and Y are, respectively, the port impedance and port admittance while δV and δF indicate the velocity errors and force errors, respectively. Since no causality needs to be assigned *a priori* in this model, a generic framework is given to describe cooperative, collaborative, and competitive interactions in a unified way. More importantly, we can observe impedance control from a new perspective. For example, when two robots fight, one can plan its impedance control strategy only depending on the surmise on what impedance the other will appear at the interaction port.

Of course, when either force errors or velocity errors vanish, the feedback-connected interaction model may be correspondingly simplified to the case $\delta F = 0$ or $\delta V = 0$.

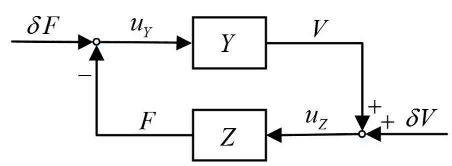


Fig. 4: The feedback-connected interaction model.

B. Interaction stability analysis

If there are both force and velocity perturbations, *i.e.*, $\delta F \neq 0$ and $\delta V \neq 0$, then there is:

Theorem 4.3: Consider the feedback interconnection $\Sigma_{Z,Y}$ with Z, $Y : \mathscr{L}_{2e}(\mathbb{R}^r) \to \mathscr{L}_{2e}(\mathbb{R}^r)$ in Fig. 4.

i). Assume that for any δF, $\delta V \in \mathscr{L}_{2e}(\mathbb{R}^r)$ there are solutions F, $V \in \mathscr{L}_{2e}(\mathbb{R}^r)$. If Z and Y are passive then $\Sigma_{Z,Y}$ with inputs $(\delta F, \delta V)$ and outputs (V, F) is passive.

ii). Assume that for any δF, $\delta V \in \mathscr{L}_{2e}(\mathbb{R}^r)$ there are solutions F, $V \in \mathscr{L}_{2e}(\mathbb{R}^r)$. If Z and Y are strictly output passive then $\Sigma_{Z,Y}$ with inputs $(\delta F, \delta V)$ and outputs (V, F) is $\mathscr{L}_2$-stable.

Proof: For any δF, $\delta V \in \mathscr{L}_{2e}(\mathbb{R}^p)$ and $T \geq 0$

$$\begin{aligned} \langle V, u_Y \rangle_T + \langle F, u_Z \rangle_T &= \langle V, \delta F - F \rangle_T + \langle F, \delta V + V \rangle_T \\ &= \langle V, \delta F \rangle_T + \langle F, \delta V \rangle_T \end{aligned} \tag{21}$$

By passivity of Z and Y, there are constants β_Y, β_z such that

$$\langle V, \delta F \rangle_T + \langle F, \delta V \rangle_T \geq -\beta_Y - \beta_Z \tag{22}$$

from which we draw Conclusion *i)* by the passivity definition.

Likewise, there are constants $\varepsilon_Y > 0$, $\varepsilon_Z > 0$, β_Y, β_z such that by strict output passivity of Z and Y

$$\begin{aligned}\langle V, \delta F\rangle_T + \langle F, \delta V\rangle_T &\geq \varepsilon_Y \|V_T\|_2^2 + \varepsilon_Z \|F_T\|_2^2 - \beta_Y - \beta_Z \\ &\geq \varepsilon(\|V_T\|_2^2 + \|F_T\|_2^2) - \beta_Y - \beta_Z\end{aligned} \tag{23}$$

where $\varepsilon = \min\{\varepsilon_Y, \varepsilon_Z\}$. From (23), we conclude that $\Sigma_{Z,Y}$ with inputs $(\delta F, \delta V)$ and outputs (V, F) is strictly output passive which in turn implies Conclusion *ii)* by Theorem 2.7. ∎

If there is only velocity perturbation, *i.e.*, $\delta F = 0$ and $\delta V \neq 0$, we have:

Theorem 4.4: Consider the feedback interconnection $\Sigma_{Z,Y}$ with Z, $Y: \mathscr{L}_{2e}(\mathbb{R}^r) \to \mathscr{L}_{2e}(\mathbb{R}^r)$ in Fig. 4. Assume that for any $\delta F = 0$ and $\delta V \in \mathscr{L}_{2e}(\mathbb{R}^r)$ there are solutions F, $V \in \mathscr{L}_{2e}(\mathbb{R}^r)$. If there are constants ε_Z, δ_Y, β_Y β_Z such that for any u_Y, $u_Z \in \mathscr{L}_{2e}(\mathbb{R}^r)$ and $T \geq 0$

$$\begin{aligned}\langle Y(u_Y), u_Y\rangle_T &\geq \delta_Y \|(u_Y)_T\|_2^2 - \beta_Y \\ \langle Z(u_Z), u_Z\rangle_T &\geq \varepsilon_Z \|(Z(u_Z)_T\|_2^2 - \beta_Z\end{aligned} \tag{24}$$

and

$$\delta_Y + \varepsilon_Z \geq 0 \tag{25}$$

then $\Sigma_{Z,Y}$ with input δV and output F is passive or $\mathscr{L}_2$-stable whenever $\delta_Y + \varepsilon_Z = 0$ or $\delta_Y + \varepsilon_Z > 0$.

Proof: Since Z has OFP(ε_Z) and Y has IFP(δ_Y), there are constants ε_Z, δ_Y, β_Z, β_Y such that

$$\begin{aligned}\langle F, u_Z\rangle_T &\geq \varepsilon_Z \|F\|_T^2 - \beta_Z \\ \langle V, u_Y\rangle_T &\geq \delta_Y \|u_Y\|_T^2 - \beta_Y = \delta_Y \|F\|_T^2 - \beta_Y\end{aligned} \tag{26}$$

Putting $u_Y = -F$ and $u_Z = \delta V + V$ into the above inequalities and adding them up, we have for any u_Z, $u_Y \in \mathscr{H}_e$ and $T \geq 0$

$$\begin{aligned}\langle V, u_Y\rangle_T + \langle F, u_Z\rangle_T &= \langle V, -F\rangle_T + \langle F, \delta V + V\rangle_T \\ &= \langle F, \delta V\rangle_T \\ &\geq (\delta_Y + \varepsilon_Z)\|F\|_T^2 - \beta_Z - \beta_Y\end{aligned} \tag{27}$$

The results are evident from the passivity definitions. ∎

Likewise, If there is only force perturbation, *i.e.*, $\delta V = 0$ and $\delta F \neq 0$, we have:

Theorem 4.5: Consider the feedback interconnection $\Sigma_{Z,Y}$ with Z, $Y: \mathscr{L}_{2e}(\mathbb{R}^r) \to \mathscr{L}_{2e}(\mathbb{R}^r)$ in Fig. 4. Assume that for any $\delta V = 0$ and $\delta F \in \mathscr{L}_{2e}(\mathbb{R}^r)$ there are solutions F, $V \in \mathscr{L}_{2e}(\mathbb{R}^r)$. If there are constants $\tilde{\varepsilon}_Y$, $\tilde{\delta}_Z$, $\tilde{\beta}_Y$ $\tilde{\beta}_Z$ such that for any u_Y, $u_Z \in \mathscr{L}_{2e}(\mathbb{R}^r)$ and $T \geq 0$

$$\begin{aligned}\langle Y(u_Y), u_Y\rangle_T &\geq \tilde{\varepsilon}_Y \|Y(u_Y)_T\|_2^2 - \beta_Y \\ \langle Z(u_Z), u_Z\rangle_T &\geq \tilde{\delta}_Z \|(u_Z)_T\|_2^2 - \beta_Z\end{aligned} \tag{28}$$

and

$$\tilde{\delta}_Z + \tilde{\varepsilon}_Y \geq 0 \tag{29}$$

then $\Sigma_{Z,Y}$ with input δV and output F is passive or $\mathscr{L}_2$-stable whenever $\tilde{\delta}_Z + \tilde{\varepsilon}_Y = 0$ or $\tilde{\delta}_Z + \tilde{\varepsilon}_Y > 0$.

Remark 4.6: The conditions (25) and (29) say that the net amount of energy dissipated must be nonnegative. When Z and Y have the complementary dissipativity properties, one of them can be nonpassive if the excess of passivity of its counterpart can compensate the shortage of passivity.

V. Robust stability of interaction

Strictly output passivity implies $\mathscr{L}_2$-stability by Theorem 2.7. Evidently, positive output feedback passivity index implies robustness of $\mathscr{L}_2$-stability. In practice, many factors such as sampling rate, quantization, computational delay, amplifier dynamics, Coulomb damping, input saturation will influence impedance following accuracy [16], [35]. One interesting question should be posed: Does interaction stability still holds when there are impedance following errors? The answer to this question amounts to whether or not output feedback passivity index remains positive under the perturbation of the closed-loop operator.

Similar to the frequently-used mechanism in system identification, a model is said to be an approximation of a real system, provided that they have the close responses to the same excitation signal. Consider an operator $\Gamma: \mathscr{H}_e \to \mathscr{H}_e$ and its nominal model Γ_a. Let $\delta y = y - y_a$ where y and y_a denote their respective outputs. It is reasonable to say that Γ_a is an approximation of Γ if there exist $\gamma_a > 0$ and $b_a \geq 0$ such that

$$\|\Gamma_a(u) - \Gamma(u)\|_T^2 \leq \gamma_a^2 \|u\|_T^2 + b_a, \quad \forall u \in \mathscr{H}_e, \quad \forall T \geq 0 \tag{30}$$

Following the analogous discussion in [33], [34], we get the passivity indices of a perturbed system.

Theorem 5.1: Assume that (30) holds and that Γ_a^{-1} has finite gain γ_η. If Γ_a has OFP($\rho > \gamma_a$), then Γ has OFP($\rho - \gamma_a$) when the following inequality holds:

$$\frac{1}{\gamma_\eta^2} + \gamma_a^2 + \left(\frac{2}{\rho} - \rho\right) - 3 \geq 0 \tag{31}$$

VI. Conclusions and Discussions

To date, interaction stability theory is not well-developed, which has been hindered the progress of interaction control. In this paper, we make four contributions. First, the bond-graph-based interaction model is recapitulated to manifest the invariance of port functions while two subsystems interact, which elucidates the working mechanism of impedance control. Second, the input-output stability with the differential operator, usually emerging in impedances/admittances, is proved. Third, based on the uncertainty modelling idea borrowed from robust control theory, the operator-based interaction model is proposed to account for nonidealized factors in interaction. Since no causality issue arises, this model furnishes a generic framework to describe cooperative, collaborative, and competitive interactions in a unified way. Fourth, three interaction stability theorems with respect to three assumptions on force and velocity perturbations are given, especially the stability conditions while one of the interacted subsystems may be nonpassive. Finally, the robustness of interaction stability is addressed while there are imepdance/admittance uncertainties.

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