

A Dual Quaternion-Based Discrete Variational Approach for Accurate and Online Inertial Parameter Estimation in Free-Flying robots

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Abstract—The performance of model-based motion control for free-flying robots relies on accurate estimation of their parameters. In this work, a method of rigid body inertial parameter estimation which relies on a variational approach is presented. Instead of discretizing the continuous equations of motion, discrete dual quaternion equations based on variational mechanics are used to formulate a linear parameter estimation problem. This method depends only on the pose of the rigid body obtained from standard localization algorithms. Recursive semi-definite programming is used to estimate the inertial parameters (mass, rotational inertia and center of mass offset) online. Linear Matrix Inequality constraints based on the pseudo-inertia matrix ensure that the estimates obtained are fully physically consistent. Simulation results demonstrate that this method is robust to disturbances and the produced estimates are at least one order of magnitude more accurate when compared to discretization using finite differences.

I. INTRODUCTION

For autonomously performing precise tasks such as docking, load transportation or motion in cluttered environments, in space, advanced model-based controllers are usually employed. This makes the accuracy of task execution dependent on system characterization. Further, inertial properties can change during operations like payload transportation, or due to fuel use. Accurate and reliable algorithms that carry out on-board parameter estimation are thus essential.

A common approach for parameter estimation of free-flying robots is to formulate the problem linearly in the parameters using the Newton-Euler equations of motion [1], [2]. This requires acceleration measurements or their integrated versions [3]. Numerically differentiating or integrating measured states amplifies noise and additional filtering steps are needed. One such pre-filtering approach is proposed in our previous work, [4]. Some other approaches use reduced order dynamics or the conservation of momentum equations which do not depend on accelerations, but the problem has to be solved non-linearly [5], or free-floating mode has to be assumed [6], [2]. A novel approach is used in [7] by employing the quaternion discrete variational integrator equations for rotational dynamics from [8] to estimate spacecraft inertia recursively. In place of discretizing the continuous-time motion equations, variational integrators are derived from the discrete action sum, which is formed with the discrete Lagrangian. These equations only use attitude measurements

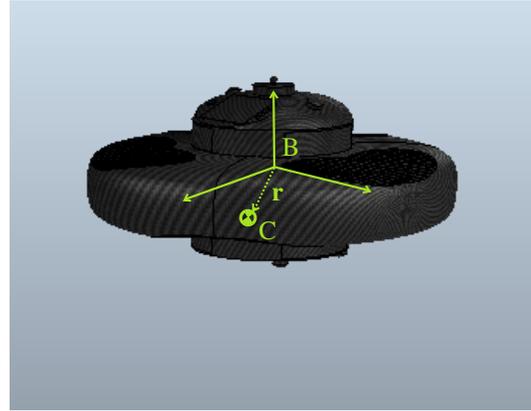


Fig. 1: Illustration of the Space CoBot free-flyer showing origin of the body frame (B), CoM (C), and offset (r).

and are linear with respect to the inertia. Moreover, integrators derived from discrete variational mechanics are known for their energy and momentum conservation properties and high-quality linearizations [9].

Inspired by [7], we formulate a full rigid body inertial parameter estimation problem by using discrete variational mechanics. We use the Lie Group variational integrators based on unit dual quaternions from [10], where they have been derived for the first time. Due to dual quaternions, the full 6DoF discrete equations of motion can be expressed without decoupling translation and rotations. The equations have been parametrized by 6 independent variables such that the state evolves in $SE(3)$. We pose this problem linearly with respect to the spatial inertia parameters, resulting in a linear optimization problem that can be solved in real-time. The estimated parameters will only define a meaningful and physically consistent system when certain constraints are satisfied. For instance, the positive definiteness of the inertia matrix and positive value of mass. When dealing with noisy measurement data that has not been pre-filtered, the best fit to the minimization problem could end up violating some of these constraints. This issue often goes unaddressed in works on inertial parameter estimation. A useful technique is to formulate the objective as a Semi-Definite Programming (SDP) problem with Linear Matrix Inequalities (LMI) constraints [7], [11]. SDPs are convex problems and a variety of fast solvers are available to find the solution (if one exists) very efficiently in real-time. Moreover, several standard problems like linear and quadratic programming can be cast in SDP form [12]. In our work, we use SDP with the physical-consistency LMIs that involve a pseudo-inertia matrix defined in [11] to characterize

*This work was supported by FCT project [UID/EEA/50009/2019] and P2020 INFANTE 10/SI/2016. The authors are grateful to these sponsors.

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physical plausibility and resolve this issue. Therefore, the contributions of this paper are as follows:

- A 6DoF rigid body parameter estimation problem using discrete equations of motion based on principles of variational mechanics is formulated.
- The aforementioned parameter estimation problem is converted to an SDP framework and solved while imposing physical consistency LMI constraints using the pseudo-inertia matrix from [11].
- The use of this method is shown to improve estimation accuracy by at least one order of magnitude as compared to results obtained from finite difference approximation, even in the presence of noise (fig. 5).

To the best of our knowledge, this is the first time that discrete equations based on the principles of variational mechanics have been used for full body inertial parameter estimation.

The paper is organized as follows: Section II lays down the background on spatial body inertia and variational integrators. Dual quaternion basics and relevant algebra is also presented. Section III develops on these concepts and reviews the dual quaternion variational equations. Then, the optimization problem for calculating the inertial parameters is formulated. Recursive Semi-Definite Programming with physical consistency Linear Matrix Inequality constraints are proposed as an approach for estimating these parameters. Section IV presents the implementation details of this approach in simulation, and Section V presents the results.

II. BACKGROUND

A. The spatial body inertia

Let C be the Center of Mass (CoM) of the robot and B the origin of the body-fixed frame, expressed in the inertial frame. We also define $B - C = \mathbf{r}$ as the CoM offset from the body frame origin, expressed in the body frame. This is illustrated in fig. 1. We use the spatial vector notation, which combines the angular and linear quantities of aspects of rigid-body motion as one 6D vector. Let $\mathbf{V}_B = [\boldsymbol{\omega}_B^T \mathbf{v}_B^T]^T$ be the spatial velocity of the body, with $\boldsymbol{\omega}_B$ being its angular velocity about an axis passing through B and \mathbf{v}_B , the linear velocity of a point on the body coinciding with B . The kinetic energy of a rigid body can be written as [13]:

$$T = \frac{1}{2} \mathbf{V}_B^T \mathbf{M} \mathbf{V}_B \quad (1)$$

Here, \mathbf{M} is the 6×6 spatial inertia matrix

$$\mathbf{M} = \begin{bmatrix} J_B & mS(\mathbf{r}) \\ mS^T(\mathbf{r}) & m\mathbf{I}_{3 \times 3} \end{bmatrix} \quad (2)$$

where $J_B = J_C + mS(\mathbf{r})S^T(\mathbf{r})$. J_B and J_C are the rotational inertias of the body about B and C . $S(\mathbf{r})$ represents the vector \mathbf{r} written in skew-symmetric matrix form. We have considered B to be the origin of the body-fixed frame, but it could be any reference point.

Hereon J_B will be written simply as J . Due to the definition of the spatial body inertia, the motion of a rigid

body with non-coinciding CoM and body frame can be represented in a compact notation. This is also useful for applications where the robot has a shifted CoM due to grasped payload.

B. Variational Integrators

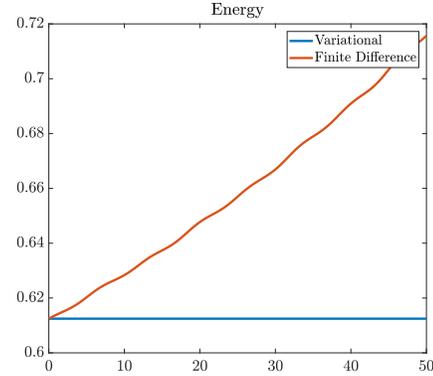


Fig. 2: Change in energy of a free-rigid body

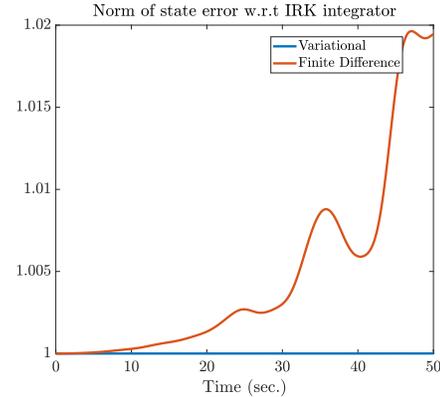


Fig. 3: Errors in state evolution with respect to the Runge-Kutta integrator, for the case of variational and finite difference integrators. The errors in quaternion are calculated as $q_2^\dagger \circ q_1$, as a result the norm of state error is 1 at time $t = 0$.

Instead of discretizing the continuous motion equations, the discrete-time equations (discrete variational integrators) are obtained by applying the principles of variational mechanics to the discrete action sum [14]. These integrators have the advantages of being computationally efficient and stable. They also yield numerical solutions that are faithful to the continuous equations of motion. Moreover, they are symplectic and known for their energy and momentum conservation properties. Lie group variational integrators are a subclass of variational integrations which discretize the equations of motion for rigid bodies [15], [16], [17]. These integrators conserve the geometric structure of the system as it evolves on the Lie Group without the need for local maps or projection. It is to be noted that general methods of numerical integration like the Range-Kutta do not explicitly preserve the characteristics of the configuration space. These properties makes the discrete variational equations desirable for accurate online estimation applications.

The energy conservation and linearization accuracy of the discrete variational equations are illustrated with the help of figs. 2 and 3. The simulated system is a 6DoF free rigid body meaning that the energy should be conserved. Fig. 2 illustrates the energy change in the system simulated using the Variational integrator (VI)¹ as compared to continuous equations discretized using the finite difference (FD) method². Fig. 3 shows the change in states for a free body for the DQVI and the FDI as compared to an Implicit Runge-Kutta Gauss-Legendre 4 integrator (IRK - GL4). Implicit RK methods have a higher order of accuracy, good stability properties, and perform better than explicit methods for simulation of real-world problems [18]. The evolution of the states obtained from the DQVI closely follows that of the IRK while the states obtained through the FDI drifts overtime.

C. Dual quaternions

Dual quaternions (DQ) are singularity free, unambiguous, as well as the most compact $SE(3)$ rigid transform representations [19]. Unlike quaternions, they can represent position and attitude of a rigid body. A dual quaternion is a vector composed of two quaternions which form its real and dual part. That is, for a DQ $\tilde{\mathbf{q}} \in \mathbb{R}^8$, $\tilde{\mathbf{q}} = \bar{\mathbf{q}}_a + \epsilon \bar{\mathbf{q}}_b$, where ϵ is the dual operator, $\epsilon^2 = 0$, $\epsilon \neq 0$. In this article, we use tilde to denote dual quaternions and the overhead bar to denote a quaternion. The product of two dual quaternions $\tilde{\mathbf{d}}_1 = [\bar{\mathbf{d}}_{1a}^T \ \bar{\mathbf{d}}_{1b}^T]^T$ and $\tilde{\mathbf{d}}_2 = [\bar{\mathbf{d}}_{2a}^T \ \bar{\mathbf{d}}_{2b}^T]^T$ is

$$\tilde{\mathbf{d}}_1 \otimes \tilde{\mathbf{d}}_2 = \bar{\mathbf{d}}_{1a} \circ \bar{\mathbf{d}}_{2a} + \epsilon (\bar{\mathbf{d}}_{1a} \circ \bar{\mathbf{d}}_{2b} + \bar{\mathbf{d}}_{2a} \circ \bar{\mathbf{d}}_{1b}) \quad (3)$$

The notation \otimes denotes dual quaternion multiplication and \circ denotes quaternion multiplication.

DQs that represent rigid body translation and rotation transformations satisfy the unit conditions of $\|\tilde{\mathbf{q}}\| = 1$ and $\bar{\mathbf{q}}_a^\dagger \circ \bar{\mathbf{q}}_b + \bar{\mathbf{q}}_b \circ \bar{\mathbf{q}}_a^\dagger = 0$, where \dagger represents quaternion conjugate.

III. PROBLEM FORMULATION

A. Dual Quaternion Variational Integrators

Dual quaternions and their algebra relevant for this article were summarized in II-C. The discrete variational form of the full body equations of motion is derived in reference [10] and summarized in this section.

Let \mathbf{l} be the position of a rigid body and $\bar{\mathbf{q}}$ represent its attitude in unit quaternions ($\mathbf{l} \in \mathbb{R}^3$, $\bar{\mathbf{q}} \in \mathbb{R}^4$ and $\|\bar{\mathbf{q}}\| = 1$) Let $\tilde{\mathbf{p}}$ be a unit DQ used to represent rigid body pose,

$$\tilde{\mathbf{p}} = \bar{\mathbf{q}} + \epsilon \frac{1}{2} \hat{\mathbf{l}} \circ \bar{\mathbf{q}} \quad (4)$$

where $\hat{\mathbf{l}}$ signifies a pure quaternion

$$\hat{\mathbf{l}} = \begin{bmatrix} \mathbf{l} \\ 0 \end{bmatrix} \quad (5)$$

The attitude in quaternions is related to the angular velocity of a rigid body as $\hat{\omega}_B = 2\bar{\mathbf{q}}^\dagger \circ \dot{\bar{\mathbf{q}}}$. A similar relationship

¹The VI is the Dual Quaternion Variational Integrator (DQVI) from [10]

²For quaternion propagation, discrete time equations from [20] were used

exists between the pure dual quaternion $\tilde{\chi}_B$ of the velocities expressed in the body frame, and the pose dual quaternion $\tilde{\mathbf{p}}$.

$$\tilde{\chi}_B = \begin{bmatrix} \hat{\omega}_B \\ \hat{\mathbf{v}}_B \end{bmatrix} \quad (6)$$

$$\tilde{\chi}_B = 2\tilde{\mathbf{p}}^\dagger \otimes \dot{\tilde{\mathbf{p}}} \quad (7)$$

The symbol \dagger denotes the dual quaternion conjugate. $\dot{\tilde{\mathbf{p}}}_k$ can be approximated by using the trapezoidal rule over a fixed time step $h \approx t_{k+1} - t_k$ as:

$$\dot{\tilde{\mathbf{p}}}_k = (\tilde{\mathbf{p}}_{k+1} - \tilde{\mathbf{p}}_k) / h \quad (8)$$

A quantity $\tilde{\mathbf{f}}_k$ is defined as change in the dual quaternions between each time step,

$$\tilde{\mathbf{f}}_k = \tilde{\mathbf{p}}_k^\dagger \otimes \tilde{\mathbf{p}}_{k+1} \quad (9)$$

Considering a micro-gravity environment and the absence of conservative fields, the Lagrangian will be equal to the kinetic energy, $L = T$. Using (1) and (6), T can be expressed with velocities in the dual quaternion form and matrix M from (2) augmented with zeros as

$$T = \frac{1}{2} \tilde{\chi}_B^T M_{8 \times 8} \tilde{\chi}_B \quad (10)$$

$$M_{8 \times 8} = \begin{bmatrix} J_B & \vdots & mS(\mathbf{r}) & \vdots \\ \dots & 0 & \dots & 0 \\ mS^T(\mathbf{r}) & \vdots & mI_{3 \times 3} & \vdots \\ \dots & 0 & \dots & 0 \end{bmatrix}$$

A discrete approximation of the Lagrangian is the starting point for deriving the discrete dynamic equations of a rigid body. External forces and torques are incorporated in the variational equations by using a discrete approximation of the virtual work integral and applying Lagrange-D'Alembert's principle. The forces and torques are written in their pure dual quaternion form as $\tilde{\tau}_{Bk} = \hat{\tau}_{ak} + \epsilon \hat{\tau}_{bk}$. Here τ_{ak} are the applied torques and τ_{bk} are the forces expressed in body frame. Moreover, Φ and $\Psi \in \mathbb{R}^3$ are used as independent and unconstrained vector representations for the unit dual quaternion

$$\tilde{\mathbf{f}}_k = \bar{\mathbf{f}}_a + \epsilon \bar{\mathbf{f}}_b \text{ as [7], [10]}$$

$$\bar{\mathbf{f}}_a = \begin{bmatrix} \sqrt{1 - \Phi^2} \\ \Phi \end{bmatrix}, \quad \bar{\mathbf{f}}_b = \begin{bmatrix} -\frac{\Psi \cdot \Phi}{\sqrt{1 - \Phi^2}} \\ \Psi \end{bmatrix} \quad (11)$$

It is clear that $|\Phi| < 1$ for these parameters to be valid. Following the derivation in [10], the discrete rigid body dynamic equations in dual quaternion form are the following.

$$\begin{cases} \mathcal{A}(\Phi_k, \Psi_k) - \alpha_k = 0 \\ \mathcal{B}(\Phi_k, \Psi_k) - \beta_k = 0 \end{cases} \quad (12)$$

$$\begin{aligned} \mathcal{A}(\Phi_k, \Psi_k) = & \left(-\frac{\Psi_k \cdot \Phi_k}{\sqrt{1 - \Phi_k^2}} I + S(\Psi_k) \right) (M_{21} \Phi_k + M_{22} \Psi_k) \\ & + \left(\sqrt{1 - \Phi_k^2} I + S(\Phi_k) \right) (M_{11} \Phi_k + M_{12} \Psi_k) \end{aligned} \quad (13)$$

$$\begin{aligned} \alpha_k = & \left(\sqrt{1 - \Phi_{k-1}^2} I - S(\Phi_{k-1}) \right) (M_{11} \Phi_{k-1} + M_{12} \Psi_{k-1}) \\ & - \left(\frac{\Psi_{k-1} \Phi_{k-1}}{\sqrt{1 - \Phi_{k-1}^2}} I + S(\Psi_{k-1}) \right) (M_{21} \Phi_{k-1} + M_{22} \Psi_{k-1}) \\ & + \frac{h^2}{2} \tau_{ak} \end{aligned} \quad (14)$$

$$B(\Phi_k, \Psi_k) = \left(\sqrt{1 - \Phi_k^2} I + S(\Phi_k) \right) (M_{21} \Phi_k + M_{22} \Psi_k) \quad (15)$$

$$\begin{aligned} \beta_k = & \left(\sqrt{1 - \Phi_{k-1}^2} I - S(\Phi_{k-1}) \right) (M_{21} \Phi_{k-1} + M_{22} \Psi_{k-1}) \\ & + \frac{h^2}{2} \tau_{bk} \end{aligned} \quad (16)$$

where M_{ij} for i and $j = 1, 2$ are the 3×3 matrices that form the spatial inertia matrix M from (1)

B. Parameter Estimation Problem

We define matrix \mathbf{J} and vector operator G

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} J_{xx} & J_{yy} & J_{zz} & J_{xy} & J_{xz} & J_{yz} \end{bmatrix}^T \\ G(\Phi) &= \begin{bmatrix} \Phi_1 & 0 & 0 & \Phi_2 & \Phi_3 & 0 \\ 0 & \Phi_2 & 0 & \Phi_1 & 0 & \Phi_3 \\ 0 & 0 & \Phi_3 & 0 & \Phi_1 & \Phi_2 \end{bmatrix} \end{aligned} \quad (17)$$

Using (2) and (17) we can write:

$$\begin{aligned} M_{11} \Phi &= J_B \Phi = G(\Phi) \mathbf{J} \\ M_{12} \Phi &= mS(r) \Phi = -mS(\Phi) \mathbf{r} \\ M_{21} \Phi &= mS^T(r) \Phi = mS(\Phi) \mathbf{r} \\ M_{22} \Phi &= mI_{3 \times 3} \Phi = m\Phi \end{aligned} \quad (18)$$

Additionally, the following matrices are defined:

$$\begin{aligned} P_k &= \left(-\frac{\Psi_k \Phi_k}{\sqrt{1 - \Phi_k^2}} I - S(\Psi_k) \right) \\ Q_k &= \left(-\frac{\Psi_k \Phi_k}{\sqrt{1 - \Phi_k^2}} I + S(\Psi_k) \right) \\ X_k &= \left(\sqrt{1 - \Phi_k^2} I - S(\Phi_k) \right) \\ Y_k &= \left(\sqrt{1 - \Phi_k^2} I + S(\Phi_k) \right) \end{aligned} \quad (19)$$

Let $S(\Phi_k)$ be denoted as \mathfrak{S}_k . Using (18) and (19), (13) - (16) can be written as

$$\begin{aligned} \mathcal{A}(\Phi_k, \Psi_k) &= mQ_k (\mathfrak{S}_k \mathbf{r} + \Psi_k) \\ &+ Y_k (G_k \mathbf{J} - m\mathfrak{S}_k \mathbf{r}) \\ \alpha_k &= X_{k-1} (G_{k-1} \mathbf{J} - m\mathfrak{S}_{k-1} \mathbf{r}) \\ &+ mP_{k-1} (\mathfrak{S}_{k-1} \mathbf{r} + \Psi_{k-1}) + \frac{h^2}{2} \tau_{ak} \\ B(\Phi_k, \Psi_k) &= mY_k (\mathfrak{S}_k \mathbf{r} + \Psi_k) \\ \beta_k &= mX_{k-1} (\mathfrak{S}_{k-1} \mathbf{r} + \Psi_{k-1}) + \frac{h^2}{2} \tau_{bk} \end{aligned} \quad (20)$$

We substitute (20) in (12). On writing the equations in matrix form, we get:

$$\begin{bmatrix} A_k & B_k & C_k \\ 0_{3 \times 6} & D_k & E_k \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ m\mathbf{r} \\ m \end{bmatrix} = -\frac{h^2}{2} \begin{bmatrix} \tau_{ak} \\ \tau_{bk} \end{bmatrix} \quad (21)$$

$$\begin{aligned} A_k &= Y_k G_k - X_{k-1} G_{k-1} \\ B_k &= (Q_k - Y_k) \mathfrak{S}_k + (X_{k-1} - P_{k-1}) \mathfrak{S}_{k-1} \\ C_k &= Q_k \Phi_k - P_{k-1} \Phi_{k-1} \\ D_k &= (Y_k \mathfrak{S}_k - X_{k-1} \mathfrak{S}_{k-1}) \\ E_k &= Y_k \Psi_k - X_{k-1} \Psi_{k-1} \end{aligned} \quad (22)$$

Finally, (21) can be written in a compact form,

$$H_k \boldsymbol{\pi} = T_k \quad (23)$$

where

$$\begin{aligned} H_k &= \begin{bmatrix} A_k & B_k & C_k \\ 0_{3 \times 6} & D_k & E_k \end{bmatrix} \\ \boldsymbol{\pi} &= \begin{bmatrix} \mathbf{J}^T & m\mathbf{r}^T & m \end{bmatrix}^T \\ T_k &= -\frac{h^2}{2} \begin{bmatrix} \tau_{ak} \\ \tau_{bk} \end{bmatrix} \end{aligned} \quad (24)$$

Consider the matrix subscript \underline{N} to denote that the matrix is composed of all measurements from time step 1 to N

$$H_{\underline{N}} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \quad T_{\underline{N}} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}$$

If noise (\mathbf{v}) on the measured torques (\tilde{T}) is assumed to be zero mean Gaussian, then the Maximum Likelihood Estimator (MLE) for the observation model

$$\tilde{T} = H\boldsymbol{\pi} + \mathbf{v} \quad (25)$$

is equivalent to the least squares solution. Consider independent and identically distributed noise on the measured torques, we can write the problem as

$$\text{minimize}_{\boldsymbol{\pi}} \frac{1}{2} \|H_{\underline{N}} \boldsymbol{\pi} - T_{\underline{N}}\|_2^2 \quad (26)$$

This problem can be solved for unknown $\boldsymbol{\pi}$ in batch for data from time 1 to N using the standard least squares solution, provided that matrix H is full-rank and well-conditioned.

$$\boldsymbol{\pi} = (H_{\underline{N}}^T H_{\underline{N}})^{-1} H_{\underline{N}}^T T_{\underline{N}} \quad (27)$$

As more measurement data arrives, the dimensions of the problem would increase rapidly. One way of solving the least squares problems efficiently and recursively is to use matrix decompositions [21]. Here, like in [7], we use the QR decomposition (or Gram-Schmidt) [22] of matrix H . The $m \times n$ matrix H is split as the product of an orthonormal $m \times m$ matrix Q and an upper triangular $m \times n$ matrix R . As $m > n$, rows from $n + 1$ to m of R will be zero.

$$H = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} \quad (28)$$

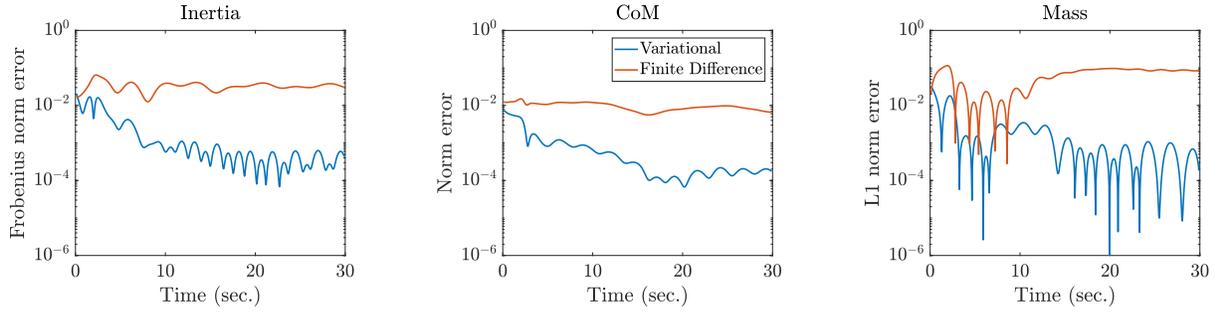


Fig. 4: Relative estimation errors obtained through the two approaches

The *economy* QR decomposition (denoted here as $\mathbf{QR}_{\text{econ}}$) only computes matrices Q_1 and R . Since Q is orthonormal, the objective to minimize from (26) becomes $R\pi - z$, where $Q^T T = z$. At every time step, the matrices are updated as:

$$Q_{N+1}, R_{N+1} = \mathbf{QR}_{\text{econ}} \left(\begin{bmatrix} R_N \\ H_{N+1} \end{bmatrix} \right) \quad (29)$$

$$z_{N+1} = Q_{N+1}^T \begin{bmatrix} z_N \\ T_{N+1} \end{bmatrix} \quad (30)$$

The objective at each step then reduces to

$$\underset{\pi}{\text{minimize}} \quad \frac{1}{2} \|R_N \pi - z_N\|_2^2 \quad (31)$$

Note that the dimension of R and z remains the same for every step, and yet the information from new measurements is taken into account.

C. Semi-Definite Programming and physical consistency

A 6D spatial inertia matrix is physically semi-consistent if it satisfies the constraints of $m > 0$, $J_C \succ 0$, i.e, the mass and rotational inertia are positive definite. Further, the triangular inequalities must be satisfied in order to make sure that the estimated rotational inertias are density realizable: $J_{C11} + J_{C22} \geq J_{C33}$, $J_{C22} + J_{C33} \geq J_{C11}$ and $J_{C11} + J_{C33} \geq J_{C22}$. This makes the spatial inertia matrix to be fully physically consistent [11].

The least squares problem in (27) is converted to a semi-definite program (SDP) (section 1.1 of [12]), such that the estimates can be constrained. SDPs are convex optimization problems with constraints expressed as Linear Matrix Inequalities (LMI). From [11], all the physical consistency constraints can be combined into one Linear Matrix Inequality (LMI) constraint on the pseudo-inertia matrix, which we call \underline{M} . Enforcing the positive definiteness of \underline{M} ensures the application of all the above constraints.

$$\underline{M} \succ 0, \quad \underline{M} = \begin{bmatrix} \Sigma & m\mathbf{r} \\ m\mathbf{r}^T & m \end{bmatrix} \quad (32)$$

$$\Sigma = \frac{1}{2} \text{Tr}(J) I_{3 \times 3} - J \quad (33)$$

Converting the recursive optimization from (31) to a recursive SDP problem, we get

$$\underset{\pi'}{\text{minimize}} \quad \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi \\ s \end{bmatrix}$$

$$\text{subject to} \quad \begin{cases} \begin{bmatrix} s & (R_N \pi - z_N)^T \\ (R_N \pi - z_N) & \mathbf{I} \end{bmatrix} \succeq 0 \\ \underline{M} \succ 0 \end{cases}$$

This problem is solved at every time step with updated values of R and z . Here, s is a slack variable. The SDP has a linear objective, and the earlier objective of (31) is expressed as a convex constraint. Usually, all the LMI constraints are expressed in a tiled-manner as one diagonal matrix. However, in this case, the positive definiteness constraint on the 4×4 pseudo inertia matrix \underline{M} replaces the individual physical consistency constraints, which gives us computational benefits while solving this problem recursively.

IV. IMPLEMENTATION

The robot model was simulated with the following inertial parameters: $m = 5$ kg, $J(J_B)_{xx} = 1.5$, $J_{yy} = 2$, $J_{zz} = 3$, $J_{xy} = 0.1$, $J_{xz} = 0.2$, $J_{yz} = 0.3 \text{ kgm}^2$ and r_x, r_y and r_z as 0.05, 0.1 and 0.05 m respectively. Fast integrator of the type Implicit Runge-Kutta Gauss Legendre 4 (IRK-GL4) exported by the ACADO toolkit [25] was used to simulate the 6 Degree of Freedom rigid body dynamics. Time steps of 0.05s were used. For the sake of simplicity, we use sinusoidal waves with different phase and amplitudes to provide actuation. However, the trajectories can be optimized for information gain to ensure that the regressor (matrix H_N from (26)) is full rank and well-conditioned, as in our previous work, [4].

We consider that the states consisting of position, attitude and linear and angular velocities are obtained through a standard pose determination algorithm. For comparison with the proposed method, a finite difference (FD) based discretization approach was used to approximate the angular and linear accelerations. Similar to the discrete variational case, the FD equations can be arranged linear to π (The derivation can be found in our previous work, [4]). The parameters are then estimated by solving the SDP with CVX [23] [24] in MATLAB. On a 4 core Intel i5-7200U CPU @ 2.50GHz running Ubuntu 16.04 LTS, CVX took on average 0.2s to solve each constrained least squares problem using SDP. However, it should be noted that the problem can in principle be engineered in such a way that real time performance can be attained.

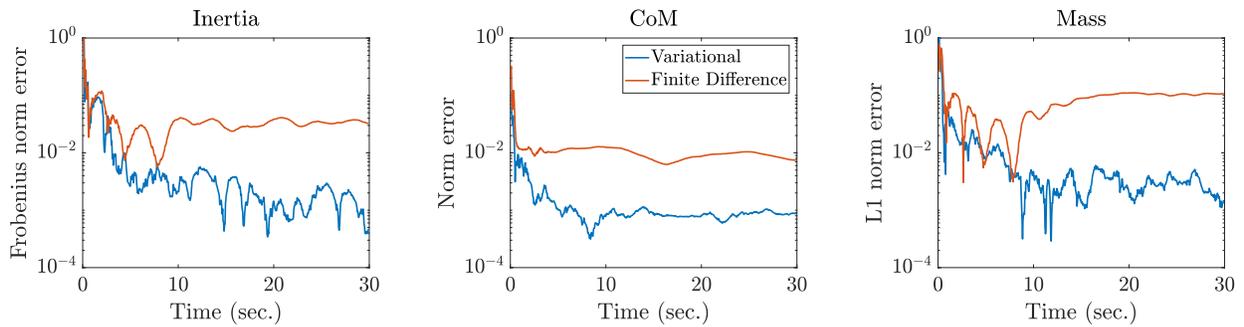


Fig. 5: Relative estimation errors when torques and velocities were corrupted with zero mean Gaussian noise

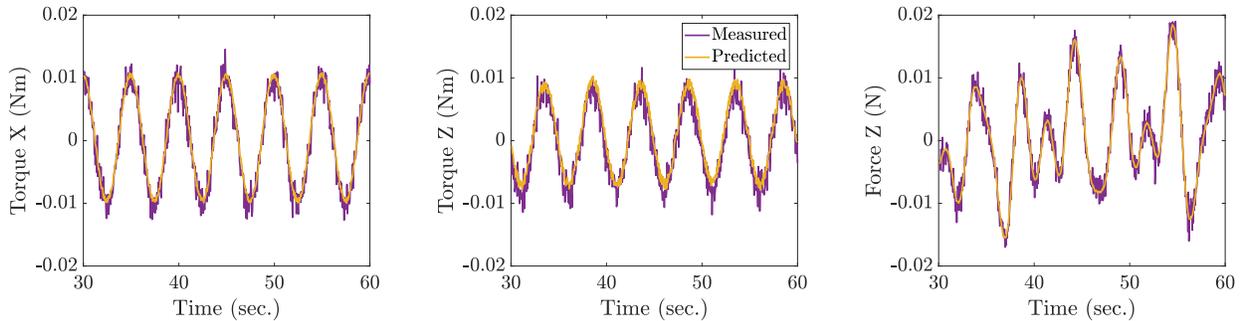


Fig. 6: Regression errors for the next 30 secs. of simulation. Parameter estimates obtained at $t = 30$ were used.

	Inertial parameters										Accuracy relative to FD		
	J_{xx}	J_{yy}	J_{zz}	J_{xy}	J_{xz}	J_{yz}	r_x	r_y	r_z	m	Inertia	CoM offset	Mass
1	2.08	3.19	4.06	-0.0487	0.0739	-0.0655	-0.08	0.06	0.15	10	2.09×10^1	1.20×10^1	2.18×10^2
2	5	5	5	0	0	0	0.1	0.2	0.1	5	1.92	3.05	2.01×10^1
3	0.0453	0.0417	0.0519	0	0	0	0	0	0	6.047	1.06	2.42	9.12

TABLE I: Estimation accuracy due to the proposed method as a multiple of the accuracy obtained with the finite difference method. The relative accuracy is calculated as $\frac{\text{error}_{\text{FD}}}{\text{error}_{\text{V}}}$, for inertia, CoM and mass. (FD: finite difference, V: variational)

V. RESULTS

Fig. 4 presents the results of parameter estimation with recursive least squares in the case of no added noise. The estimation errors with the variational method for each of the parameters are greater than one order of magnitude less than those obtained through the finite difference equations.

Further, the measured forces and torques were corrupted with zero mean Gaussian disturbances of spectral density $10^{-3} \text{ units}/\sqrt{\text{Hz}}$. The velocities in simulation were also corrupted with process noise of $10^{-5} \text{ units}/\sqrt{\text{Hz}}$. Fig. 5 clearly shows that after just a few samples in the simulation, estimates yielded by the proposed method are more accurate than the finite difference discretization method. As the end of 30 seconds, the estimates obtained using the proposed method are at least one order of magnitude more accurate. For the next 30 seconds, the measured states and estimated parameters were used to calculate the residual errors (forces and torques predicted as $\hat{T}_N = H_N \hat{\pi}$). Three of them are plotted in fig. 6 along with their measured values. Finally, estimation results for three other sets of inertial parameters with noisy measurements are given in table I. The benefit of using the discrete variational approach for full body inertial parameter estimation, i.e, ten parameters can be clearly seen.

VI. CONCLUSION

This paper presented an inertial parameter estimation algorithm for a free-flying robot based on the discrete variational motion equations and unit dual quaternions. This method does not require estimates of the accelerations. Further, the spatial inertia matrix and dual quaternion representation makes it possible to write the full body discrete variational equations in a compact and tractable manner. The desired parameters are made to appear linearly with respect to the dynamic properties. Then, Semi Definite Programming with physical consistency Linear Matrix Inequality constraints is used to obtain meaningful estimates. This method was tested in simulation for cases with and without noise. Even in the presence of noise, the proposed approach yielded estimates with at least one order of magnitude more accurate than those of the finite difference method of discretization. This difference is important for the accuracy of motion algorithms dependent on the system model. In the future, we plan to validate the approach through real experiments using free-flyers on board the International Space Station (ISS), as well as study higher-order symplectic methods and their impact on the accuracy of the estimates.

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